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<tr><td>Partner organization</td><td colspan="2">Politehnica University of Timisoara</td></tr>
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<tr><td>Lesson title</td><td colspan="2">Improper Integrals</td></tr>
<tr><td>Learning objectives</td><td>Definition of an improper integrals
Basic idea and Example
Geometric interpretation of an area for to the improper integrals
Example of divergence of an improper integrals
Convergence criteria for the improper integrals</td><td rowspan="4">Methodology
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<tr><td>Aim of the lecture / Description of the practical problem</td><td>The problem: If the interval is infinite won’t the area be infinite? Now one might initially assume that if the interval of integration is infinite, the area under the curve must also be infinite, but actually this is not always the case.</td></tr>
<tr><td>Previous knowledge assumed:</td><td>Knowledge of Integration
Knowledge of the properties of Riemann’s integral</td></tr>
<tr><td>Introduction / Theoretical basics</td><td>The students are asked about their research on primitives (homework).
The solution of the practical problem.</td></tr>
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According to the description of the problem given in section "Description of the practical problem", we have the following solution:

Take for example the curve one over x squared and we want to evaluate $\int_1^{\infty} \frac{1}{x^2} dx$.

Question for the students:

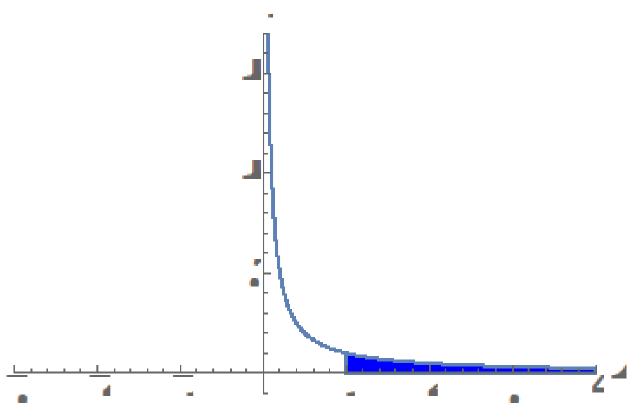
- How we evaluate this integral?

We can treat this like a definite integral and get the antiderivative

$$\int_1^t \frac{1}{x^2} dx = \left. \frac{-1}{x} \right|_1^t = 1 - \frac{1}{t}$$

Task for the students:

- For different values of t what happens to the area under the graphic?



We can observe that if we give larger and larger values for t then we get closer and closer to one and if t is infinity, which means integrating over this whole portion of the function all the way to infinity, then we get one as the answer.

Task for the students:

- According to the previous remark which is the area under the curve from one to infinity?

Assessment of learning

☐Test
☐Quiz
☑Presentation
☑Project
☐Published work

That means that the area under the curve from x equals one all the way to infinity is one.

This gives us a simple way to evaluate improper integrals. We can integrate f of x dx from a to infinity and this will simply be equal to the integral over the definite interval from a to t in the limit of t approaching infinity:

$$\lim_{t\longrightarrow\infty}\int_a^t f(x)dx$$

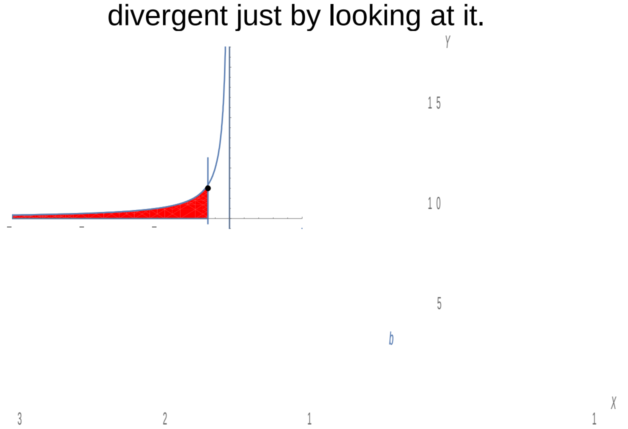
This limit may or may not exists, which means that not all improper integrals can be evaluated, but some indeed can.

After solving the task, the next question is set for the students:

- What happens in the negative infinity direction to b for the integral

$$\lim_{t\rightarrow-\infty}\int_t^b f(x)dx\ ?$$

If we are evaluating an improper integral and we do get a finite number an answer, then we say that the integral is convergent. If the limit does not exists meaning the area truly is infinite, we say that the integral is divergent. Unfortunately, it may be difficult to tell whether an integral is convergent or divergent just by looking at it.



Action	• Definition and basic concepts The concept of Riemann integrability in effect, familiar reader since the last class of high school, can expand the functions defined on unbounded intervals and also to unbounded functions: $f:[a,b)\longrightarrow R$, with $b=\infty$, or b is a finite number, but f is an unbounded function, respectively $f:(a,b]\longrightarrow R$, with $a=-\infty$, or a is a finite number, but f is an unbounded function. Questions for the students: • Define an improper integral! • What do you observe? Let f:[a,b)⟶R be an integrable function on any interval [a,u]\subset [a,b). In this case it makes sense to consider the function F:[a,b) ⟶R, $F(u)\overset{def}{=}\int_a^u f(x)dx$ **Definition:** The function f is integrable in the generalized sense on [a,b) if and only if exists $\lim_{n\longrightarrow\infty} F(u)$ (as a finite number). In this case we will say that the improper integral $\int_a^b f(x)dx$ is convergent and we define: $$\int_a^b f(x)dx\overset{def}{=}\lim_{\substack{u\to b\\ u<b}}\int_a^u f(x)dx$$ The extremity b of the definition interval (can be infinity or in which neighborhood f is an unbounded function) is a singular point of the improper integral $\int_a^b f(x)dx$. If the singular point minus infinity or infinity then the improper integral is easy to recognize; the singular point to finite distance is often highlighted by the suggestive notation	

$\int_a^{b-0} f(x)dx$, or $\int_{a+0}^{b} f(x)dx$.

Remark: The following results will be presented only for functions f:[a,b)⟶R, because all of which can immediately reformulate for the functions f:(a,b]⟶R by simply change of variables x ⟶ -x in the case of unbounded interval, respectively x⟶ a+b-x in the case in which the function is unbounded.

Outstanding improper integrals

- $\int_0^{\infty} e^{-mx}dx=\frac{1}{m}, m>0$
- $\int_a^{\infty} \frac{1}{x^p}dx\,is \begin{cases} convergent, if\ p\in(1,\infty) \\ divergent, if\ p\in \text{¿} \end{cases}, a>0$
- $\int_a^{b-0} \frac{1}{(b-x)^p}dx\,is \begin{cases} convergent, if\ p\in(0,1) \\ divergent, if\ p\in \text{¿} \end{cases},$ a<b
- $\int_{a+0}^{b} \frac{1}{(x-a)^p}dx\,is \begin{cases} convergent, if\ p\in(0,1) \\ divergent, if\ p\in \text{¿} \end{cases},$ a<b

Task for the students:

- According to the previous integrals, we want to evaluate in case by convergence.

Convergence Criteria of Improper Integrals

3. **A Comparison Test for Improper Integrals**

Let f,g:[a,b)⟶R, be an integrable function on [a,u]⊂[a,b), ∀ u: a<u<b, f(x)≤g(x), x∈[a,b).

If $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)dx$ is convergent.

If $\int_a^b f(x)dx$ is divergent then $\int_a^b g(x)dx$ is divergent.

4. **Practical Convergence Criteria**

- If f:[a,∞) $\longrightarrow R_+$ is integrable on [a,u]⊂ [a,∞), and $\lim\limits_{x\longrightarrow\infty} x^{p} f(x)$ is finite, and nonzero with $p\in(1,\infty)$ then $\int\limits_a^{\infty} f(x)dx$ is convergent.
- If f:[a,b) $\longrightarrow R_+$ is integrable on [a,u]⊂ [a,b), and $\lim\limits_{\substack{x\longrightarrow b\\ x<b}} (b-x)^{p} f(x)$ is finite, and nonzero with $p\in(0,1)$ then $\int\limits_a^{b-0} f(x)dx$ is convergent
- If f:(a,b] $\longrightarrow R_+$ is integrable on [a,u]⊂ (a,b], and $\lim\limits_{\substack{x\longrightarrow a\\ x>a}} (x-a)^{p} f(x)$ is finite, and nonzero with $p\in(0,1)$ then $\int\limits_{a+0}^{b} f(x)dx$ is convergent.

Remark. Conclusions on convergence remain valid even if the limit in the hypothesis is zero. Using any of the three variants of practical criteria for to conclude divergence of improper integrals (i.e. $p\in[0,1]$in the first case, $p\in[1,\infty)$ in the other two cases) is possible only if the limits is nonzero.

Task for the students:

- Study the convergence of the following integrals, and if is possible evaluate:

$$\int\limits_0^8 \frac{1}{\sqrt[3]{8-x}}dx;$$

$$\int_{3}^{\infty} \frac{1}{x^2+5x-6}\,dx.$$

Example 1. *Determine if the following integrals are convergent or divergent, using the definition*:

1. $\displaystyle\int_{-1}^{7} \frac{1}{\sqrt[3]{1+x}}\,dx$;
2. $\displaystyle\int_{-\infty}^{1} \frac{1}{x+1}$dx

Solution: i) $\displaystyle\int_{-1}^{7} \frac{1}{\sqrt[3]{1+x}}\,dx = \lim_{\substack{u\to -1\\ u>-1}} \int_{u}^{7} \frac{1}{\sqrt[3]{1+x}}\,dx$

$\displaystyle = \lim_{\substack{u\to -1\\ u>-1}} \frac{3}{2}(1+x)^{\frac{2}{3}}\Big|_{u}^{7} = 6$, so the integral is convergent.

ii) $\displaystyle\int_{-\infty}^{1} \frac{1}{x+1}$dx= $\displaystyle\lim_{u\to -\infty} \int_{u}^{1} \frac{1}{x+1}\,dx$=

= $\displaystyle\lim_{u\to -\infty} \ln|x+1|\Big|_{u}^{1} = -\infty$, so the integral is divergent.

Example 2. Using the comparison criteria to limit, to show that the integral $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{x}{\cos x}\,dx$ is divergent.

Solution. We know that $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{\pi-2x}\,dx = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\,dx$ is divergent. We evaluate

$$l=\frac{\pi}{2}\lim_{\substack{x\to\frac{\pi}{2}\\ x>\frac{\pi}{2}}}\frac{\frac{x}{\cos x}}{\frac{1}{\pi-2x}}=\frac{\pi}{2}\lim_{\substack{x\to\frac{\pi}{2}\\ x>\frac{\pi}{2}}}\frac{\pi-2x}{\cos x}=\pi.$$

We conclude that the both integrals are of the same nature, thus the integral $\int_0^{\frac{\pi}{2}}\frac{x}{\cos x}dx$ is divergent.

Example 3. Determine whether the following integrals are convergent, using the practical criteria of convergence:

i) $\int_{-\infty}^{0}\frac{1-3x}{\sqrt{x^6+x^3+1}}dx$; ii) $\int_{0}^{\sqrt{3}}\frac{x+1}{\sqrt{(3-x^2)^3}}dx$.

Solution.

i) $\lim_{x\to-\infty}(-x)^p\frac{1-3x}{\sqrt{x^6+x^3+1}}=$

$=\lim_{x\to-\infty}\frac{x^3\left(\frac{1}{x}-3\right)}{x^3\vee\sqrt{1+\frac{1}{x^3}+\frac{1}{x^6}}}=3, for\ p=2>1$, so the integral $\int_{-\infty}^{0}\frac{1-3x}{\sqrt{x^6+x^3+1}}dx$ is convergent.

ii) $\lim_{\substack{x\to\sqrt{3}\\ x>\sqrt{3}}}(\sqrt{3}-x)^p\frac{x+1}{\sqrt{(3-x^2)^3}}=$

$=\frac{\lim_{\substack{x\to\sqrt{3}\\ x>\sqrt{3}}}\sqrt{(\sqrt{3}-x)^3}(x+1)}{\sqrt{(\sqrt{3}-x)^3(\sqrt{3}+x)^3}}=\frac{\sqrt{3}+1}{\sqrt{24\sqrt{3}}}$, for $p=\frac{3}{2}>1$,

so the integral $\int_{0}^{\sqrt{3}}\frac{x+1}{\sqrt{(3-x^2)^3}}dx$ is divergent.

<table>
<tr><td>Materials / equipment / digital tools / software</td><td colspan="2"><u>The materials for learning:</u> the references at the end of the document;
<u>Equipment</u>: classroom, blackboard/whiteboard, different colours of chalk/markers;
<u>Digital tools</u>: laptop, projector, smart board;
<u>Software</u>: Wolfram Mathematica, Geogebra.</td></tr>
<tr><td>Consolidation</td><td colspan="2">• Use of materials, equipment, digital tools, software by teachers and students;
• The teacher's discussion with the students through appropriate questions;
• Independent solving of simple tasks by the students under the supervision of the teacher;
• Given of examples by the teacher for introducing a new concept in a cooperation and a discussion with the students;
• Assignment of homework by the teacher with a time limit until the next class.</td></tr>
<tr><td colspan="3">Reflections and next steps</td></tr>
<tr><td colspan="2">Activities that worked</td><td>Parts to be revisited</td></tr>
<tr><td colspan="2">After the class, the teacher, according to his personal perceptions regarding the success of the class, fills in this part.</td><td>Based on the homework done by the students, and on the questions and discussion at the beginning of the next class, the teacher concludes which parts of this class should be revised.</td></tr>
<tr><td colspan="3">References</td></tr>
<tr><td colspan="3">[1] D. Paunescu, A. Juratoni –Calcul integral avansat, Editura Orizonturi Universitare, Timisoara 2015</td></tr>
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